NONLINEAR METHOD OF VEHICLE VELOCITY DETERMINATION BASED ON INVERSE SYSTEM AND TENSOR PRODUCT OF LEGENDRE POLYNOMIALS - INTERMEDIATE CLASS

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Abstract

Presented paper discusses two different nonlinear approaches to precrash velocity determination for vehicles from Intermediate Car Class. Data that was used to perform analyses introduced in this paper was taken from National Highway Traffic Safety Administration (NHTSA) database. Such database is comprised of substantial number of crash cases and main focus was put on frontal impacts.

Hitherto used energy methods are based on linear model which proves to be inaccurate and producing significant errors. Presented considerations concern the inverse system and tensor product of Legendre polynomials. The focus of those methods is to establish the value of nonlinear coefficient b_k which is the slope factor of precrash velocity $\mathrm{V_t}$ and deformation ratio $\mathrm{C_s}$ function.

Keywords: Location, manufacture, protection, VIN, Vehicle Identification Number, vehicle originality

Nomenclature:
EES – Equivalent Energy Speed [m/s]
NHTSA – National Highway Traffic Safety Administration

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Cs – deformation ratio [m]
C_1–C_6 – deformation coefficients
E/M – specific collision energy [J/kg]
L_t – dent zone width [m]
V_t – vehicle speed [m/s]
W_{def} – work of deformation [J]
b_k – constant slope factor [m/s/m]
b_{sg} – limit speed [m/s]
m – weight of car [kg]
n – number of cases [-]
error – EES estimation error [-]
A [N/m], B [N/m²], G [N], C [m], α [m], β [m²] – coefficient

1. Introduction

In recent years the development of methods for precrash velocity determination seemed to reach a plateau. Linear models, however simple and showing great ease of use, unfortunately produce significant inaccuracies, reaching up to 30% [39]. Authors focus on developing superior methods in terms of precision, based on nonlinear models, which already have shown promising results in the recent past [18-26].

Presented paper discusses another two nonlinear approaches to precrash velocity determination in Intermediate Car Class. These considerations concern the inverse system and tensor product of Legendre polynomials [1,3]. The focus of those methods is to establish the value of nonlinear coefficient b_k which is the slope factor. The pre-impact velocity V_t is a function of said coefficient and deformation ratio C_s [4,8-12,16-23,20,21,25-28,30,34-37,39]. The C_s determines the body deformation as an arithmetic mean of deformation depth [23] in specific points C_1 to C_6 [13-15,23,26,34,35,37,39].

Once the above parameters are established, the deformation work can be defined. This quantity is then directly used to find the value of EES from the following equation:

$$EES = \sqrt{\frac{2W}{m}} \tag{1}$$

where: EES Equivalent Energy Speed, W deformation work, m vehicle mass.

The EES [6,39,39] defines the amount of energy to deform the vehicle [7,24,38,32,33] while impacting a rigid obstacle, i.e. only plastic deformations occur and the entire kinetic energy accumulated by the vehicle is transferred to deformation work. This is then used as a baseline to find the actual velocity of a vehicle in a real life accident and asses if, for example, collision avoiding maneuvers has been applied [2,23] or if the vehicle speed was properly adjusted according to traffic conditions [34]. Analyzed set of cases focuses on frontal impacts [35,36].

The first method of EES determination is based on a simple observation, that coefficient b_k dependent on C_s behaves similarly to function $\rightarrow \frac{1}{x}$. This led Authors to consideration of inverse systems: 1, $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$. Application of the least square approximation gives much better results than other methods used in this field.

The second method focuses on Legendre polynomials, which are orthogonal function in the domain of [–1,1]. For the purpose of Octave implementation, the set of Legendre polynomials was rescaled and renumbered. Finally, the Legendre polynomials tensor product is considered and the least square method is applied.

2. Inverse system method description

Assuming that there are given points $(x_n, y_n)_{n=1}^{N}$ and a function set $(f_m)_{m=1}^{M}$. The goal is to find coefficients $(a_m)_{m=1}^{M}$, that minimize the value of:

$$\sum_{n=1}^{N}\left(y_n - \sum_{m=0}^{M} a_m f(x_n)\right)^2 \tag{2}$$

This problem is called the least-square function approximation and is very well covered in the literature [2]. The first author of this solution was not established, since both Carl Gauss and Adriena-Marie Legendre were working on the topic.

In order to find the minimum, the function has to be differentiated with respect to a_k, $k = 1, \ldots, M$ obtaining a series of equations

$$\forall_{k=1,\ldots,M} \sum_{m=1}^{M} a_m \left(\sum_{n=1}^{N} f_k(x_n) f_m(x_n)\right) = \sum_{n=1}^{N} y_n f_k(x_n) \tag{3}$$

It is convenient to use the matrix notation:

$$\begin{pmatrix} \sum_{n=1}^{N} f_1(x_n) f_1(x_n) & \cdots & \sum_{n=1}^{N} f_1(x_n) f_M(x_n) \\ \vdots & \ddots & \vdots \\ \sum_{n=1}^{N} f_M(x_n) f_1(x_n) & \cdots & \sum_{n=1}^{N} f_M(x_n) f_M(x_n) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_M \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N} y_n f_1(x_n) \\ \vdots \\ \sum_{n=1}^{N} y_n f_M(x_n) \end{pmatrix} \tag{4}$$

It is worth noticing that matrix M on LHS (2) is symmetric in sense, that $M(k,m) = M(m,k)$ for arbitrary k, $m = 1, \ldots M$. This obvious observation allows to speed up the calculations almost twice. In that stage, the solution of least square problem is down to solving of the M matrix and making following observations, on condition that it is inverse:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_M \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N} f_1(x_n) f_1(x_n) & \cdots & \sum_{n=1}^{N} f_1(x_n) f_M(x_n) \\ \vdots & \ddots & \vdots \\ \sum_{n=1}^{N} f_M(x_n) f_1(x_n) & \cdots & \sum_{n=1}^{N} f_M(x_n) f_M(x_n) \end{pmatrix}^{-1} \begin{pmatrix} \sum_{n=1}^{N} y_n f_1(x_n) \\ \vdots \\ \sum_{n=1}^{N} y_n f_M(x_n) \end{pmatrix} \tag{5}$$

Returning to the original problem of finding the b_k as a function of C_s, the set of points $(x_n,y_n)_{n=1}^{N}$ is adjacent to $(C_s(n), bk(n))_{n=1}^{N}$, where n is the number of crash tests. As shown in the preceding sections, the following is true:

$$f_1(x) = 1,\ f_2(x) = \frac{1}{x},\ f_3(x) = \frac{1}{x^2}\ \ and\ \ f_4(x) = \frac{1}{x^3}. \tag{6}$$

The solution has the following form:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} N & \sum_{n=1}^{N} \frac{1}{C_s(n)} & \sum_{n=1}^{N} \frac{1}{C_s^2(n)} & \sum_{n=1}^{N} \frac{1}{C_s^3(n)} \\ \sum_{n=1}^{N} \frac{1}{C_s(n)} & \sum_{n=1}^{N} \frac{1}{C_s^2(n)} & \sum_{n=1}^{N} \frac{1}{C_s^3(n)} & \sum_{n=1}^{N} \frac{1}{C_s^4(n)} \\ \sum_{n=1}^{N} \frac{1}{C_s^2(n)} & \sum_{n=1}^{N} \frac{1}{C_s^3(n)} & \sum_{n=1}^{N} \frac{1}{C_s^4(n)} & \sum_{n=1}^{N} \frac{1}{C_s^5(n)} \\ \sum_{n=1}^{N} \frac{1}{C_s^3(n)} & \sum_{n=1}^{N} \frac{1}{C_s^4(n)} & \sum_{n=1}^{N} \frac{1}{C_s^5(n)} & \sum_{n=1}^{N} \frac{1}{C_s^6(n)} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{n=1}^{N} b_k(n)1 \\ \sum_{n=1}^{N} b_k(n)\frac{1}{C_s(n)} \\ \sum_{n=1}^{N} b_k(n)\frac{1}{C_s^2(n)} \\ \sum_{n=1}^{N} b_k(n)\frac{1}{C_s^3(n)} \end{pmatrix} \tag{7}$$

Whereas the function, that best approximates given points is of following form:

$$f(C_s) = a_1 + \frac{a_2}{C_s} + \frac{a_3}{C_s^2} + \frac{a_4}{C_s^3} \tag{8}$$

The rest of the procedure of precrash velocity determination is standard and was already discussed in [22], [24]. As a quick reminder, the subsequent coefficients are defined (with emphasis on crash test number n dependence):

$$A(n) = m(n) \cdot b_{gs} \cdot \frac{f(C_s(n))}{L_t(n)} \tag{9}$$

$$B(n) = m(n) \cdot \frac{f^2(C_s(n))}{L_t(n)} \tag{10}$$

$$G(n) = \frac{A^2(n)}{2 \cdot B(n)} \tag{11}$$

$$\alpha(n) = C_1(n) + C_6(n) + 2 \cdot \left(C_2(n) + C_3(n) + C_4(n) + C_5(n)\right), \tag{12}$$

$$\begin{aligned}\beta(n) = {} & C_1^2(n) + C_6^2(n) + 2 \cdot \left(C_2^2(n) + C_3^2(n) + C_4^2(n) + C_5^2(n)\right) + C_1(n) \cdot C_2(n) + \\ & + C_2(n) \cdot C_3(n) + C_3(n) \cdot C_4(n) + C_4(n) \cdot C_5(n) + C_5(n) \cdot C_6(n),\end{aligned} \tag{13}$$

where m is the vehicle mass, b_{gs} = 3.05 $\frac{m}{s}$ is the speed limit and L_t is the width of dent zone. This yields following relations:

$$W(n) = \frac{L_t(n)}{5}\left(A(n)\cdot\frac{\alpha(n)}{2} + B\cdot\frac{\beta(n)}{6} + 5\cdot G\right) \tag{14}$$

and

$$EES(n) = \sqrt{\frac{2\cdot W(n)}{m(n)}} \tag{15}$$

The relative error is then calculated by following formula:

$$Error = \left|\frac{V_t(n) - EES(n)}{V_{t(n)}}\right| \tag{16}$$

3. Inverse system method results

The database used in this considerations consists of 465 crash tests. Authors have used 80% of the database, and then the results are validated. For 80% the following values are given:

$$a_1 = 0.984709, a_2 = 13.705967, a_3 = -1.218736, a_4 = 0.041163 \tag{17}$$

therefore,

$$b_k(C_s) = 0.984709 + \frac{13.705967}{C_s} + \frac{-1.218736}{C_2^2} + \frac{0.041163}{C_s^3}\left[\frac{1}{s}\right] \tag{18}$$

Figure 1 presents the approximation of the inverse method.

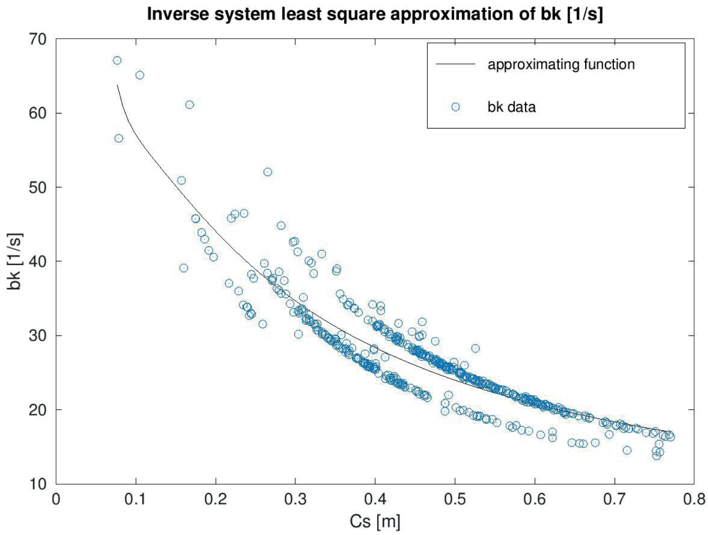

Figure 1. Inverse system least square approximation of b_k $[\frac{1}{s}]$.

For comparison, Figure 2 presents the same data range, but with linear approximation.

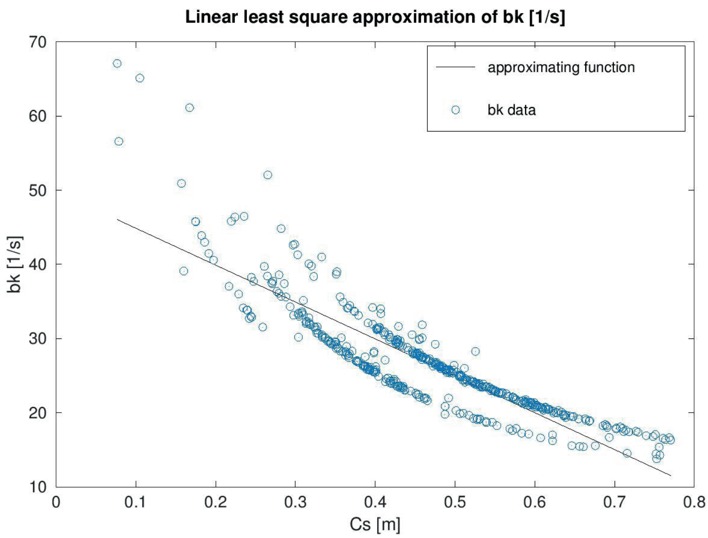

Figure 2. Linear least square approximation of b_k $[\frac{1}{s}]$.

$$b_k(C_s) = 49.654340 - 49.740533 \cdot C_s \tag{19}$$

There is no doubt that the nonlinear model is the superior one. Figures 3 and 4 present the relative error for nonlinear and linear method respectively.

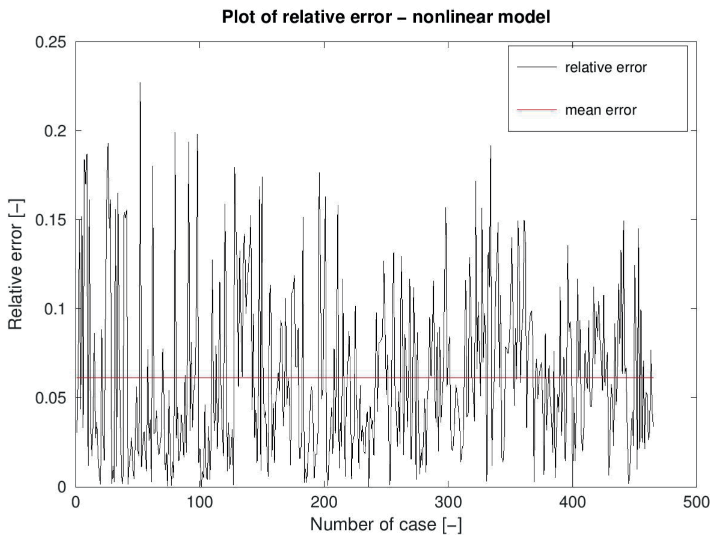

Figure 3. Relative error of nonlinear model (inverse system).

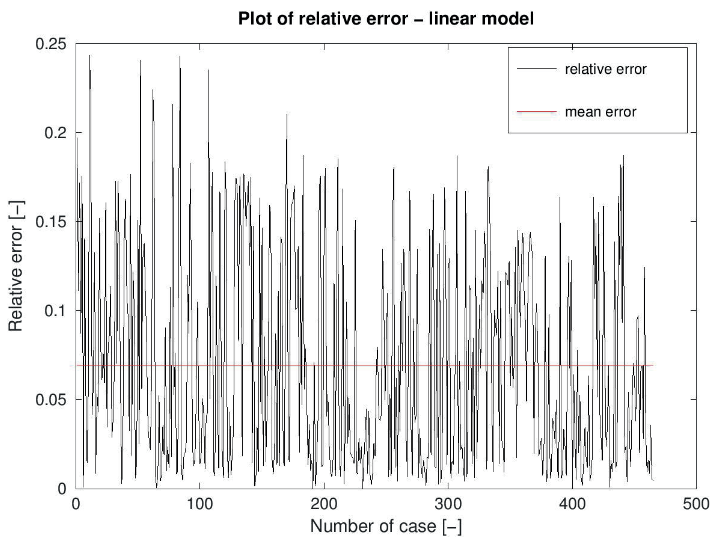

Figure 4. Relative error of linear model.

The average error of the inverse method equals to 6.3355% whereas, the linear model produces average error of 7.2675%.

Finally, a plot comparing the nonlinear and linear models is presented in Figure 5.

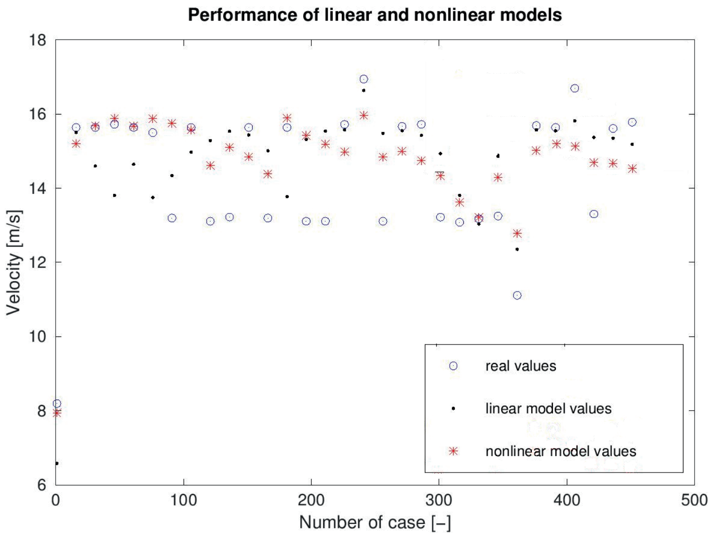

Figure 5. Performance of linear and nonlinear models (inverse system).

The detailed numerical data is enclosed in the Table 1.

Table 1. Detailed numerical values for inverse system method.

m	L_t	C_s	α	β	Linear error	Nonlinear error
1737	1943	0.076665	0.766650	0.088234	0.196983	0.030512
1685	1786	0.530300	5.303000	4.224565	0.008652	0.027690
1485	1549	0.638900	6.389000	6.129849	0.066540	0.002980
1619	1486	0.688300	6.883000	7.130574	0.121735	0.010160
1529	1351	0.635800	6.358000	6.076609	0.063555	0.002527
1542	1554	0.690900	6.909000	7.169166	0.112872	0.024523
1693	1666	0.656100	6.561000	6.461549	0.087242	0.193653
1524	1562	0.608900	6.089000	5.579293	0.042444	0.004111
1760	1595	0.427300	4.273000	2.749615	0.166005	0.114473
1650	1539	0.510300	5.103000	3.910029	0.175258	0.142013
1510	1725	0.374340	3.743400	2.424926	0.012968	0.050547

m	L_t	C_s	α	β	Linear error	Nonlinear error
1638	1321	0.397600	3.976000	2.374232	0.137785	0.090136
1615	1524	0.690200	6.902000	7.176259	0.119101	0.016266
1785	1605	0.572000	5.720000	4.938079	0.168352	0.176746
1772	1600	0.523200	5.232000	4.127745	0.185307	0.158408
1561	1465	0.479400	4.794000	3.488553	0.009353	0.047231
1753	1524	0.475200	4.752000	4.181716	0.018107	0.057765
1535	1321	0.465400	4.654000	3.250729	0.180548	0.132069
1497	1600	0.489000	4.890000	3.600990	0.007361	0.042738
1732	1570	0.440200	4.402000	2.947347	0.018997	0.062144
1581	1524	0.386100	3.861000	2.260915	0.129440	0.084313
1776	1451	0.313400	3.134000	1.480716	0.055415	0.041542
1558	1401	0.270900	2.709000	1.139417	0.010811	0.003042
1756	1500	0.377600	3.776000	2.174620	0.121413	0.078657
1602	1500	0.245220	2.452200	0.920680	0.111937	0.113721
1732	1750	0.486300	4.863000	3.580805	0.007422	0.031907
1674	1525	0.522000	5.220000	4.113652	0.005737	0.017776
1595	1525	0.455300	4.553000	3.350781	0.052643	0.098311
1612	1401	0.435100	4.351000	2.868393	0.155315	0.122781
1604	1372	0.431500	4.315000	2.822406	0.016728	0.045653
1778	1549	0.404500	4.045000	2.509753	0.037858	0.062735

4. Tensor product method description

Let us assume that there are given points $(x_n, y_n, z_n)_{n=1}^{N}$ and function family $(h_m)_{m=1}^{M}$ (functions of two variables). Again, the objective is to obtain the coefficients $(a_m)_{m=1}^{M}$, which minimize its value.

$$\sum_{n=1}^{N}\left(z_n - \sum_{m=1}^{M} a_m h(x_n, y_n)\right)^2 \tag{20}$$

Similarly to section 2, the issue of least square approximation is reduced to a linear solution:

$$\begin{pmatrix} \sum_{n=1}^{N} h_1(x_n, y_n) h_1(x_n, y_n) & \cdots & \sum_{n=1}^{N} h_1(x_n, y_n) h_M(x_n, y_n) \\ \vdots & \ddots & \vdots \\ \sum_{n=1}^{N} h_M(x_n, y_n) h_1(x_n, y_n) & \cdots & \sum_{n=1}^{N} h_M(x_n, y_n) h_M(x_n, y_n) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_M \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N} y_n h_1(x_n) \\ \vdots \\ \sum_{n=1}^{N} y_n h_M(x_n) \end{pmatrix} \tag{21}$$

As for the choice of function $(h_m)^M_{n=1}$, Legendre polynomial product tensors are chosen. Consideration involves a sequence of polynomials (P_m) defined by a iterative formula:

$$\forall_m \geq 1 (m+1)P_{m+1}(x) = (2m+1)x \cdot P_m(x) - nP_{m-1}(x) \tag{22}$$

where $P_0(x) = 1$ and $P_1(x) = x$ These are Legendre polynomials from a range of [–1,1]. First Legendre polynomials are:

$$P_0(x) = 1,\ P_1(x), P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x), \ldots \tag{23}$$

Legendre polynomials have a feature called orthogonality:

$$\forall_{i \neq j} \int_{-1}^{1} P_i(x)P_j(x)\,dx = 0 \tag{24}$$

This feature is a natural consequence of Legendre polynomials being created as a result of orthogonalization of Gram-Schmidt function family $\{1, x, x^2, x^3, \ldots\}$. Orthogonality is valuable in this case, because the matrix M on left hand side (4) is closer to diagonal matrix. This results in smaller error of coefficients $(a_m)^M_{n=1}$.

In this application, Legendre polynomials sequence is renumbered so $Q_m = P_{m-1}$. Then following is obtained:

$$\forall_{m \geq 3} (m-1)Q_m(x) = (2m-3)x \cdot Q_{m-1}(x) - (m-2)Q_{m-2}(x) \tag{25}$$

where $Q_1(x) = 1$ and $Q_2(x) = x$. To rescale the polynomials for arbitrary interval $[a,b]$, the following relation is used:

$$f_m(x) = Q_m\left(\frac{2x - a - b}{b - a}\right) \tag{26}$$

Finally, the tensor product of two function f, g is described as:

$$h(x,y) = f \otimes g(x,y) = f(x)g(y) \tag{27}$$

In this application the $(f_i)^5_{i=1}$ and $(g_j)^5_{j=1}$ are the first five Legendre polynomials. This gives 25 tensor products.

$$\begin{gathered}
h_1 = f_1 \otimes g_1,\ h_2 = f_1 \otimes g_2,\ h_3 = f_1 \otimes g_3,\ h_4 = f_1 \otimes g_4,\ h_5 = f_1 \otimes g_5, \\
h_6 = f_2 \otimes g_1,\ h_7 = f_2 \otimes g_2,\ h_8 = f_2 \otimes g_3,\ h_9 = f_2 \otimes g_4,\ h_{10} = f_2 \otimes g_5, \\
h_{11} = f_3 \otimes g_1,\ h_{12} = f_3 \otimes g_2,\ h_{13} = f_3 \otimes g_3,\ h_{14} = f_3 \otimes g_4,\ h_{15} = f_3 \otimes g_5, \\
h_{16} = f_4 \otimes g_1,\ h_{17} = f_4 \otimes g_2,\ h_{18} = f_4 \otimes g_3,\ h_{19} = f_4 \otimes g_4,\ h_{20} = f_4 \otimes g_5, \\
h_{21} = f_5 \otimes g_1,\ h_{22} = f_5 \otimes g_2,\ h_{23} = f_5 \otimes g_3,\ h_{24} = f_5 \otimes g_4,\ h_{25} = f_5 \otimes g_5,
\end{gathered}$$

5. Result of tensor product method

The database consists of 465 crash tests. Model is created upon 80% of records and then validated. The program returns following values:

$a_1 = 13.838896, a_2 = 2.748867, a_3 = -1.584037, a_4 = -0.167105, a_5 = 0.311362,$
$a_6 = -0.120319, a_7 = 0.277501, a_8 = -0.282229, a_9 = -0.110090, a_{10} = 0.555082,$
$a_{11} = -0.187698, a_{12} = 0.248710, a_{13} = -0.038370, a_{14} = -0.025183, a_{15} = 0.161465,$
$a_{16} = -0.774463, a_{17} = 1.230121, a_{18} = -2.217478, a_{19} = 0.766343, a_{20} = -0.364707$
$a_{21} = 0.442425, a_{22} = -1.410082, a_{23} = 2.156430, a_{24} = -1.706919, a_{25} = 1.989650.$

The plot of Legendre polynomials tensor product approximation is presented in Figure 6.

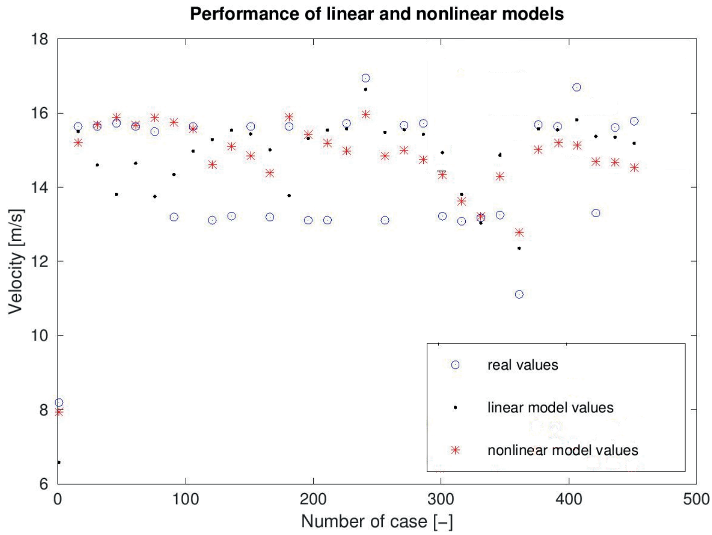

Figure 6. Tensor product approximation by orthogonal Legendre polynomials.

It is clearly visible that it has big advantage over the linear model presented in Figure 7.

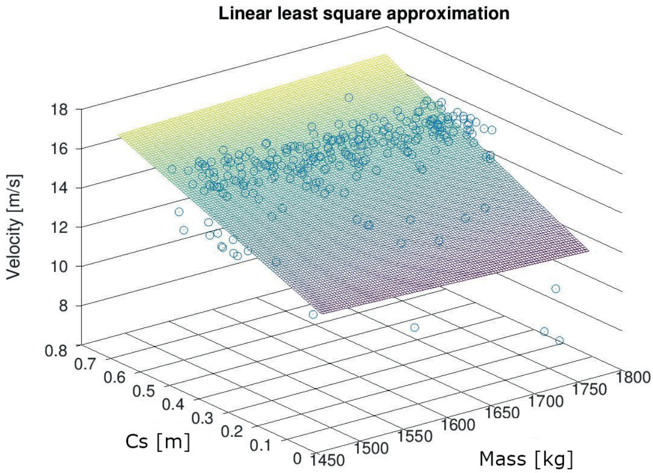

Figure 7. Linear least square approximation.

The average value of relative error is 0.058355 ≈ 5.8355% as presented in Figure 8. Compared to result from section 3, this still shows an improvement in accuracy. Moreover, it is a better result than the linear model, where the relative error was 0.069791 ≈ 6.9791% as shown in Figure 9.

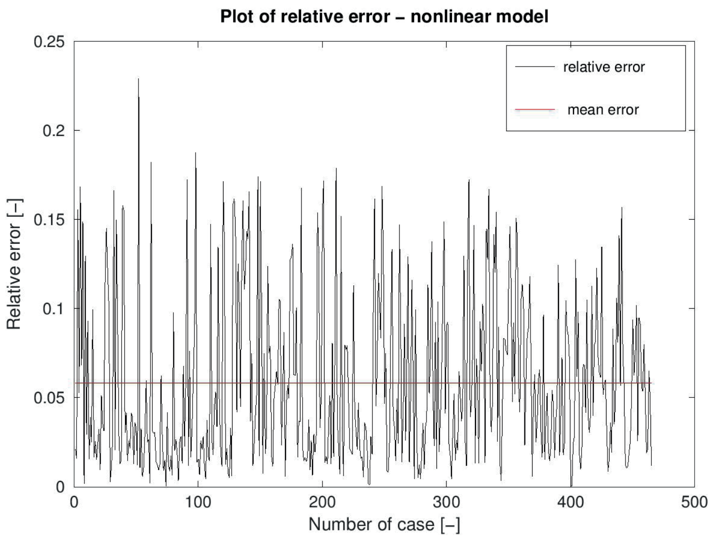

Figure 8. Value of relative error in nonlinear model.

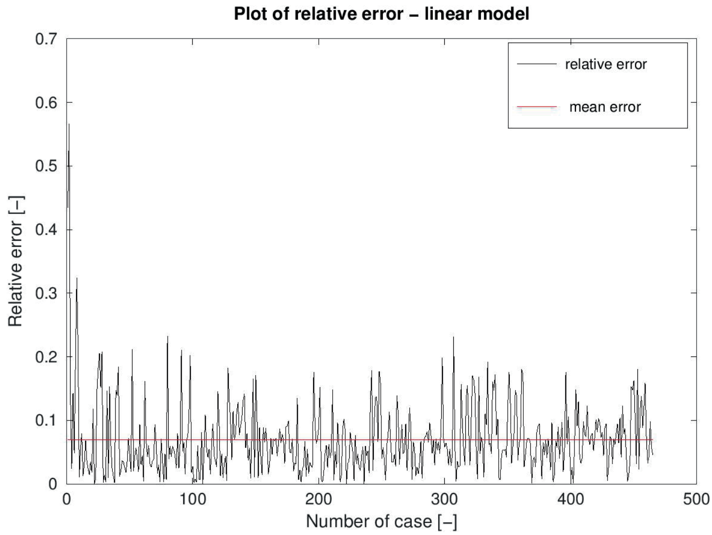

Figure 9. Value of relative error in linear model.

Finally, a comparison of linear and Legendre approach is presented in Figure 10.

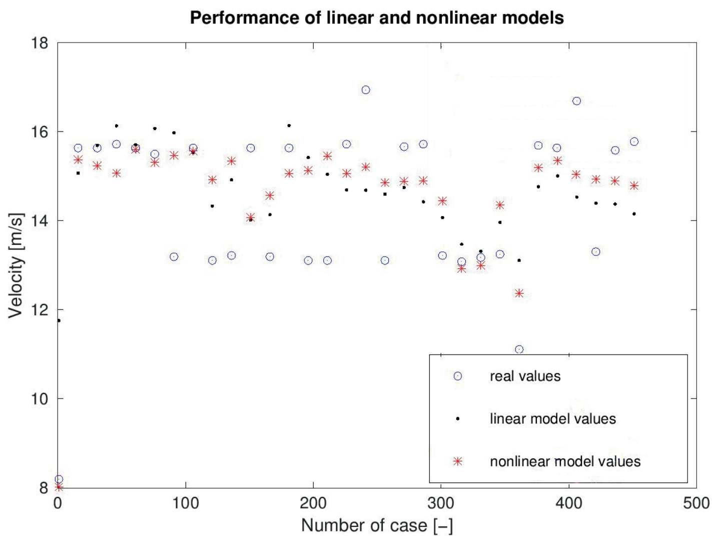

Figure 10. Performance of linear and nonlinear models (Legendre tensor product).

Table 2 presents detailed data of Legendre approach.

Table 2. Detailed numerical values of the inverse method.

m	C_s	V_t	Expected linear	Expected nonlinear	Linear error	Nonlinear error
1737	0.076665	8.194444	11.758387	8.020497	0.434922	0.021227
1685	0.530300	15.638889	15.073500	15.377113	0.036153	0.016739
1485	0.638900	15.638889	15.691242	15.240526	0.003348	0.025473
1619	0.688300	15.722222	16.134815	15.072331	0.026243	0.041336
1529	0.635800	15.638889	15.705593	15.604466	0.004265	0.002201
1542	0.690900	15.500000	16.075730	15.310253	0.037144	0.012242
1693	0.656100	13.194444	15.977276	15.469597	0.210909	0.172433
1524	0.608900	15.638889	15.527939	15.569152	0.007094	0.004459
1760	0.427300	13.111111	14.331909	14.923878	0.093112	0.138262
1650	0.510300	13.222222	14.922200	15.347597	0.128570	0.160743
1510	0.374340	15.638889	14.016845	14.078595	0.103719	0.099770
1638	0.397600	13.194444	14.137679	14.570127	0.071487	0.104262
1615	0.690200	15.638889	16.143857	15.062045	0.032289	0.036885
1785	0.572000	13.111111	15.421310	15.130071	0.176202	0.153988
1772	0.523200	13.111111	15.049170	15.456106	0.147818	0.178856
1561	0.479400	15.722222	14.695674	15.064106	0.065293	0.041859
1753	0.475200	16.944444	14.687635	15.212394	0.133189	0.102219
1535	0.465400	13.111111	14.601452	14.858930	0.113670	0.133308
1497	0.489000	15.666667	14.748927	14.889826	0.058579	0.049586
1732	0.440200	15.722222	14.429095	14.901087	0.082248	0.052228
1581	0.386100	13.222222	14.073166	14.445736	0.064357	0.092535
1776	0.313400	13.083333	13.475219	12.930332	0.029953	0.011694
1558	0.270900	13.177778	13.318723	12.999929	0.010696	0.013496
1756	0.377600	13.250000	13.963279	14.356726	0.053832	0.083526
1602	0.245220	11.111111	13.111664	12.374676	0.180050	0.113721
1732	0.486300	15.691667	14.766784	15.190986	0.058941	0.031907
1674	0.522000	15.638889	15.011052	15.360886	0.040146	0.017776
1595	0.455300	16.694444	14.536943	15.053196	0.129235	0.098311
1612	0.435100	13.305556	14.399387	14.939228	0.082209	0.122781
1604	0.431500	15.611111	14.375440	14.898420	0.079153	0.045653
1778	0.404500	15.780556	14.158808	14.790568	0.102769	0.062735

6. Conclusions

As the preceding sections clearly demonstrate, the nonlinear approach shows promising results. The advantage is best visible in the figures 6 and 7, where Authors put emphasis on the data structure itself. That is the crucial aspect when analyzing data and shows evident advantage of the nonlinear over the linear approach.

When comparing these methods to the other papers done by the Authors, it does not show such an extensive improvement. This may indicate, that this particular approach might need some further development in order to achieve comparable or even lower error values. Nevertheless, the improvement is clearly visible, especially when considering the whole spectrum of examined cases. This is due to the fact of applying spline functions, which estimate the EES speed with larger error, although the error is distributed more evenly throughout the analyzed vehicles.

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